Breaking of replica symmetry in a mean-field model of disordered membranes

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We analyze a recently introduced mean-field model of a membrane with quenched random curvature. We find the replica-symmetry-breaking solution and the equivalent Almeida-Thouless line. This line separates a flat phase from a new mixed (glassy-flat) phase with broken ergodicity. This new phase may correspond to the observed wrinkled phase of partially polymerized membranes.

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INTRODUCTION

Some partially (or incompletely) polymerized membranes seem to exhibit a low-temperature wrinkled (or wrinkled-flat) phase, characterized by *randomly frozen normals.* Examples include membranes of diacetylenic phospholipids [1,2] and of butadienic lipids [3] and possibly atom-thick silicon oxide [4] and graphite oxide [5] membranes. That phase might also describe the amorphous graphitic sheets, which have recently been shown to anneal (upon complete polymerization) into closed onionlike structures [6]. A partially polymerized membrane might be like a piece of the surface of a dried prune (wrinkled on the small scale, but flat on a large scale), in which case we shall call that state a wrinkled-flat phase: the average normal is nonzero. Or it could be similar to a crumpled piece of paper (wrinkled at all scales, i.e., average normal zero), in which case we shall refer to it as a wrinkled phase. It is important to notice that this phase is very different from the so-called crumpled phase [7] of a membrane, in which the local normals to the membrane *fluctuate* in time. It might be helpful to keep in mind the magnetic analogy where the paramagnetic, ferromagnetic, spin-glass, and mixed phases are the analog of the crumpled, flat, wrinkled, and wrinkled-flat phases, respectively.

It has been argued [8–10] that partial polymerization generates membranes with quenched-in randomness. That randomness may manifest itself in various ways. By stabilizing defects (dislocations, grain boundaries, etc.) it will appear as locally quenched inhomogeneities in the two-dimensional (2D) elastic properties of the membrane, i.e., randomness in the local metric of the membrane [9]. In certain systems, partial polymerization might also induce a random local spontaneous curvature [2]. This could be the case for diacetylenic phospholipid membranes, where the polymerized patches are similar to random roof tiles [2,11]. Such might also be the case for the atom-thick silicon oxide [4] or graphitic membranes [6], where dangling bonds (from incomplete polymerization) or local dislocations will induce a local curling of the membrane [6].

This quenched randomness, by inducting contradicting local constraints on the membrane, i.e., frustration, may be responsible for the existence of an analog of the spin-glass phase of magnetic systems, which owes its peculiar properties to the presence of frustration. Indeed, it is illuminating that the very concept of frustration was introduced [12] in the context of a two-dimensional square lattice, where each plaquette had a random discrete curvature (in internal space).

Up to now, all attempts [8–10,13,14] to assess the possible existence of a wrinkled (glassy) phase for disordered membranes, and to describe its properties, were based on a replica-symmetric (RS) *Ansatz* which is known to be an incorrect representation of the spin-glass phase of randomly frustrated magnetic systems. The purpose of this paper is to solve correctly, namely, with a Parisi *Ansatz* [15–17] [replica-symmetry breaking (RSB)], a model of randomly frustrated membrane. The model, describing a membrane with quenched random spontaneous curvature, has been introduced previously [8] and is characterized by the following Hamiltonian:

$$H = -\kappa' \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j + \sum_{\langle \alpha,\beta \rangle} V(|\mathbf{r}_\alpha - \mathbf{r}_\beta|) - \sum_{\langle i,j \rangle} \mathbf{D}'_{ij} \cdot (\mathbf{n}_i \times \mathbf{n}_j) , \tag{1}$$

where summation is over nearest neighbors. $\mathbf{r}_\alpha$ denotes the position of the αth vertex, $\mathbf{n}_i$ is the normal to the ith plaquette, κ' is the bending rigidity, $V(r)$ is a tethering potential between nearest-neighbor vertices (contributing to the elastic compression and shear moduli), and $\mathbf{D}'_{ij}$, a random vector attached to the membrane, includes a local random curvature. Hamiltonian (1) is thus rotationally invariant. In the long-wavelength limit it reduces to the usual Landau-Lifshitz elastic theory of crystalline membranes with random spontaneous curvature [8,10]. That limit was recently studied by Morse, Lubensky, and Grest [10], within a replica-symmetric *Ansatz* and by numerical simulations of Eq. (1). (In their simulations, the random vector $\mathbf{D}'_{ij}$ between two neighboring plaquettes

Δ_{ABC} and Δ_{BDC} was given by $\mathbf{D}'=a\hat{\mathbf{r}}_{AD}+b\hat{\mathbf{r}}_{BC}$, where a and b are random numbers and $\hat{\mathbf{r}}_{AD}$ and $\hat{\mathbf{r}}_{BC}$ are the unit vectors along the bonds AD and BC, respectively.) The existence of a new $T=0$ disordered flat phase has been demonstrated. For $T>0$, there is a crossover between a behavior characteristic of the $T=0$ disordered flat phase on length scales $L<L_c(T)$ (with $L_c\rightarrow\infty$ as $T\rightarrow 0$) and a behavior at scales $L>L_c$ dominated (depending on the strength of the disorder) either by the $T>0$ flat phase or by a new, possibly spin-glass phase. However, in analogy with the spin-glass problem, one does not expect the RS solution to provide a correct description of that new phase [15–17]. It would be very useful to find a replica-symmetry-breaking solution of a short-range model such as (1), but that has so far not been achieved. In the following we shall thus take the more conventional route and consider a mean-field version of Hamiltonian (1). We will present a rather detailed description of the spin-glass approach to disordered membranes, introducing the replica method and the various *Ansätze* used to solve the resulting equations. Though there are many excellent reviews [15–17] on the spin-glass problem, we shall try to be as didactic as possible.

MEAN-FIELD APPROACH

In accordance with the usual mean-field approach, we write a mean-field Hamiltonian of Eq. (1) by replacing the summation over nearest-neighbor normals with a summation over *all* possible pairs of normals (thus forfeiting all spatial information):

$$H=-\frac{\kappa'}{N}\sum_{\{i,j\}}\mathbf{n}_i\cdot\mathbf{n}_j-\sum_{\{i,j\}}\mathbf{D}'_{ij}\cdot(\mathbf{n}_i\times\mathbf{n}_j)\ . \tag{2}$$

Notice that the effect of the tethering potential [second term in Eq. (1)], which introduces long-range interactions between the normals [7], is naturally taken into account by the long-range forces implicit in the mean-field approximation. Notice also that for small deviations $f(x,y)$ of the membrane from the flat configuration, the normals in Eq. (1) are given by $\mathbf{n}\approx(-\partial_x f,-\partial_y f,1)$ and thus $\nabla\times\mathbf{n}=0$. The vectors $\mathbf{n}_i$ are thus not independent variables. This constraint is, of course, important in considering the effect of fluctuations. But within the mean-field approximation, where one is interested in obtaining the possible thermodynamic phases, the constraint may be relaxed as long as it is satisfied *a posteriori* in all phases (which is so in our case, due to the spatial independence of the mean-field solutions). Admittedly this is a drastic approximation. However, it is known [7] that the mean-field approach, Eq. (2), yields a correct quantitative description of the thermodynamic behavior of tethered membranes of inner dimensions $D>4$. The extension of the mean-field results to realistic membranes (with $D=2$) is an open problem. Due to the long-range interactions between the normals mediated by the in-plane elastic modes (tethering potential), one might hope that the mean-field phase diagram still applies qualitatively in $D=2$.

With the identification of the normals with Heisenberg spins ($\hat{\mathbf{n}}\equiv\mathbf{S}/\sqrt{3}$), our Hamiltonian (2) is identical to the mean-field Hamiltonian of a Heisenberg spin-glass [15–17] with random Dzyaloshinsky-Moriya (DM) interactions [16,18,19]. In the following we will use the spin notation with the normalization $\mathbf{S}\cdot\mathbf{S}=3$ and introduce $\kappa=\kappa'/3$ and $\mathbf{D}_{ij}=\mathbf{D}'_{ij}/3$.

The thermodynamic properties of models such as (1) and (2) are determined by the disorder-averaged free energy [15–17] $[\mathcal{F}]_{\rm av}=-k_BT[\ln Z]_{\rm av}$. The trick one uses to calculate that quantity is the replica method. Using the formula

$$\lim_{n\rightarrow 0}([Z^n]_{\rm av}-1)/n=[\ln Z]_{\rm av}=-\beta[\mathcal{F}]_{\rm av}\ , \tag{3}$$

one needs to calculate the partition function Z^n of n identical, noninteracting systems (replicas), average over the disorder, and take the limit $n\rightarrow 0$ by analytically continuing $[Z^n]_{\rm av}$ to zero. For the mean-field model (2),

$$Z^n=\int\prod_{i=1}^{N}\prod_{\alpha=1}^{n}dS_i^\alpha\exp\left[\frac{\beta}{2}\sum_{\substack{i,j\\ i\neq j}}\frac{\kappa}{N}\mathbf{S}_i^\alpha\cdot\mathbf{S}_j^\alpha+\mathbf{D}_{ij}\cdot(\mathbf{S}_i^\alpha\times\mathbf{S}_j^\alpha)\right]. \tag{4}$$

Here i indexes the spin site and α the replica. With a Gaussian probability distribution for the $\mathbf{D}_{ij}$'s,

$$P(\mathbf{D}_{ij})=\left[\frac{2\pi\Gamma^2}{N}\right]^{-3/2}\exp\left[-\frac{\mathbf{D}_{ij}^2}{2N\Gamma^2}\right], \tag{5}$$

the average over the disorder yields

$$[Z^n]_{\rm av}=\int d^3D\,P(D)Z^n=\exp[\tfrac{9}{4}nN(\beta\Gamma)^2]\int\prod_{i,\alpha}dS_i^\alpha\exp(A)\ ,$$

$$A\equiv\frac{\beta\kappa N}{2}\sum_\alpha\left[\frac{1}{N}\sum_i\mathbf{S}_i^\alpha\right]^2+\frac{N(\beta\Gamma)^2}{4}\sum_{\alpha,\beta}{}'\left\{\sum_\mu\left[\frac{1}{N}\sum_i S_{i\mu}^\alpha S_{i\mu}^\beta\right]^2-\sum_{\mu,\nu}\left[\frac{1}{N}\sum_i S_{i\mu}^\alpha S_{i\nu}^\beta\right]\left[\frac{1}{N}\sum_i S_{i\nu}^\alpha S_{i\mu}^\beta\right]\right\}$$
$$-\frac{N(\beta\Gamma)^2}{4}\sum_\alpha\sum_{\mu,\nu}\left[\frac{1}{N}\sum_i S_{i\mu}^\alpha S_{i\nu}^\alpha\right]^2, \tag{6}$$

where μ ($=x,y,z$) is the spin component and we introduce the notation $\sum'_{\alpha\beta}=\sum_{\alpha\neq\beta}$. We now use the Hubbard-Stratonovich transformation [15–17] $\exp(\pm\lambda ab)=(\lambda/2\pi i)\int dx\,dy\,\exp[\mp\lambda(xy-ax-by)]$ to express the spin traces over N sites in terms of single-site traces. Thus, for example, the sum over α in the first term in A can be written as

$$\int \prod_{i,\alpha} d\mathbf{S}_i^\alpha \exp\left[\frac{\beta\kappa N}{2}\left[\frac{1}{N}\sum_i \mathbf{S}_i^\alpha\right]^2\right]=\left[\frac{\beta\kappa N}{2\pi}\right]^{3/2}\int\int\prod_{i,\alpha} d\mathbf{m}_\alpha d\mathbf{S}_i^\alpha \exp\left[-\frac{\beta\kappa N}{2}\mathbf{m}_\alpha^2+\beta\kappa\sum_{i=1}^{N}\mathbf{m}_\alpha\cdot\mathbf{S}_i^\alpha\right]$$
$$=\left[\frac{\beta\pi N}{2\pi}\right]^{3/2}\int\prod_\alpha d\mathbf{m}_\alpha \exp\left[-\frac{\beta\kappa N}{2}\mathbf{m}_\alpha^2+N\ln\int d\mathbf{S}^\alpha e^{\beta\kappa\mathbf{m}_\alpha\cdot\mathbf{S}^\alpha}\right] \tag{7}$$

and similarly for the other terms. Next, the saddle-point approximation is used to express $[Z^n]_{\rm av}$ in the thermodynamic limit, $N\to\infty$ (at fixed $n>0$). Finally, the disorder-averaged free energy per spin, $f=[\mathcal{F}]_{\rm av}/N$, is obtained via Eq. (3) by analytically continuing $[Z^n]_{\rm av}$ to zero:

$$n\beta f=\frac{\beta\kappa}{2}\sum_\alpha \mathbf{m}_\alpha^2-\tfrac{9}{4}n(\beta\Gamma)^2-\frac{(\beta\Gamma)^2}{4}\sum_\alpha\sum_{\mu,\nu}(Q_{\mu\nu}^\alpha)^2+\frac{(\beta\Gamma)^2}{4}\sum_{\alpha,\beta}{}'\sum_{\mu,\nu}(R_{\mu\mu}^{\alpha\beta}R_{\nu\nu}^{\alpha\beta}-R_{\mu\nu}^{\alpha\beta}R_{\nu\mu}^{\alpha\beta})-\ln\int\prod_\alpha dS^\alpha\exp(-\beta\mathcal{H}_{\rm eff})\ , \tag{8}$$

where we have defined the effective Hamiltonian for n spins:

$$-\beta\mathcal{H}_{\rm eff}\equiv\beta\kappa\sum_\alpha m_\mu^\alpha S_\mu^\alpha-\frac{(\beta\Gamma)^2}{2}\sum_\alpha\sum_{\mu,\nu}Q_{\mu\nu}^\alpha S_\mu^\alpha S_\nu^\alpha+\frac{(\beta\Gamma)^2}{2}\sum_{\alpha,\beta}{}'\sum_{\mu,\nu}(R_{\mu\mu}^{\alpha\beta}S_\nu^\alpha S_\nu^\beta-R_{\mu\nu}^{\alpha\beta}S_\nu^\alpha S_\mu^\beta)\ . \tag{9}$$

The Hubbard-Stratonovich auxiliary variables m^α, $Q_{\mu\nu}^\alpha$, and $R_{\mu\nu}^{\alpha\beta}$ $(\alpha\neq\beta)$ are given by the saddle-point conditions

$$\frac{\partial f}{\partial m^\alpha}=\frac{\partial f}{\partial Q_{\mu\nu}^\alpha}=\frac{\partial f}{\partial R_{\mu\nu}^{\alpha\beta}}=0\ , \tag{10}$$

which imply that $m_\mu^\alpha=\langle S_\mu^\alpha\rangle$, $Q_{\mu\nu}^\alpha=\langle S_\mu^\alpha S_\nu^\alpha\rangle$, and $R_{\mu\nu}^{\alpha\beta}=\langle S_\mu^\alpha S_\nu^\beta\rangle$, where the angular brackets stand for a thermal average with respect to the effective Hamiltonian $\mathcal{H}_{\rm eff}$. This justifies their physical interpretation as the average magnetization, quadrupolar, and spin-glass order parameters. Due to the rotational invariance of Hamiltonian (2), we may set $Q_{\mu\nu}^\alpha=Q_\mu^\alpha\delta_{\mu\nu}$ and $R_{\mu\nu}^{\alpha\beta}=r^{\alpha\beta}\delta_{\mu\nu}+\Delta_\mu^{\alpha\beta}\delta_{\mu\nu}$ (with $\sum_\mu\Delta_\mu^{\alpha\beta}=0$). Moreover, because of the normalization $\sum_\mu Q_\mu^\alpha=3$, we let $Q_\mu^\alpha=(1+2q^\alpha,1-q^\alpha,1-q^\alpha)$.

To proceed further, we perform a cumulant expansion of Eq. (8) to $o(\beta^8)$. That expansion and the computation of the various spin traces are sketched in Appendix A. The outcome of that rather tedious calculation is

$$\beta f=\frac{\beta_\gamma\kappa_\gamma}{2}(1-\beta_\gamma\kappa_\gamma)m^2+\frac{\beta_\gamma^4\kappa_\gamma^4}{20}m^4-\tfrac{3}{2}\beta_\gamma^2(1+\tfrac{3}{5}\beta_\gamma^2)q^2-\frac{3}{2n}\beta_\gamma^2(1+\beta_\gamma^2)\sum_{\alpha,\beta}{}'(\Delta^{\alpha\beta})^2-\frac{\beta_\gamma^4\kappa_\gamma^2m^2}{n}\sum_{\alpha,\beta}{}'r^{\alpha\beta}$$
$$+\frac{3}{2n}\beta_\gamma^2(1-2\beta_\gamma^2)\sum_{\alpha,\beta}{}'(r^{\alpha\beta})^2-\frac{4\beta_\gamma^6}{n}\sum_{\alpha,\beta,\gamma}{}'r^{\alpha\beta}r^{\beta\gamma}r^{\gamma\alpha}+\frac{18\beta_\gamma^8}{5n}\sum_{\alpha,\beta}{}'(r^{\alpha\beta})^4-\frac{6\beta_\gamma^8}{n}\sum_{\alpha,\beta,\gamma,\delta}{}'r^{\alpha\beta}r^{\beta\gamma}r^{\gamma\delta}r^{\delta\alpha}+\cdots\ . \tag{11}$$

For conciseness we set $\beta_\gamma\equiv\beta\Gamma$ and $\kappa_\gamma\equiv\kappa/\Gamma$. From that expression we see that only two order parameters have a critical behavior indicative of a phase transition: the magnetization m, which becomes nonzero when $\beta\kappa>1$, and the isotropic part of the spin-glass order parameter $r^{\alpha\beta}$, which becomes nonzero when $\beta\Gamma>1/\sqrt{2}$. Notice that the anisotropic part $\Delta_\mu^{\alpha\beta}$ has no critical behavior. Thus, in contrast with the Heisenberg spin glass [15–17] ($\mathbf{D}_{ij}=0$ and $\kappa=\kappa_{ij}$ random), there is no transverse spin-glass (Gabay-Toulouse [20]) phase in our model. That is a result of the strong coupling [19] between the longitudinal (parallel to the mean magnetization) and transverse spin components, which is absent in the Heisenberg model.

REPLICA-SYMMETRIC PHASE DIAGRAM

The replica symmetric *Ansatz assumes* that the spin-glass order parameter is symmetric under a permutation of all the replicas, so that $R_{\mu\nu}^{\alpha\beta}=R_{\mu\nu}=(r+\Delta_\mu)\delta_{\mu\nu}$. Taking the limit $n\to 0$ in Eq. (11) [with $\sum'_{\alpha,\beta}(r^{\alpha\beta})^2=n(n-1)r^2$, etc.] the RS free energy is then

$$\beta f_{\rm RS}=\frac{\beta_\gamma\kappa_\gamma}{2}(1-\beta_\gamma\kappa_\gamma)m^2+\frac{\beta_\gamma^4\kappa_\gamma^4}{20}m^4-\tfrac{3}{2}\beta_\gamma^2(1+\tfrac{3}{5}\beta_\gamma^2)q^2+\tfrac{3}{2}\beta_\gamma^2(1+\beta_\gamma^2)\Delta^2+\beta_\gamma^4\kappa_\gamma^2m^2r-\tfrac{3}{2}\beta_\gamma^2(1-2\beta_\gamma^2)r^2-8\beta_\gamma^6r^3+\tfrac{162}{5}\beta_\gamma^8r^4+\cdots\ . \tag{12}$$

The saddle-point conditions, Eq. (10), imply

$$0=\beta\frac{\partial f_{\rm RS}}{\partial m}=\beta_\gamma\kappa_\gamma(1-\beta_\gamma\kappa_\gamma)m+\frac{\beta_\gamma^4\kappa_\gamma^4}{5}m^3+2\beta_\gamma^4\kappa_\gamma^2mr\ ,$$
$$0=\beta\frac{\partial f_{\rm RS}}{\partial r}=\beta_\gamma^4\kappa_\gamma^2m^2-6\beta_\gamma^2(\tfrac{1}{2}-\beta_\gamma^2)r-24\beta_\gamma^6r^2+\tfrac{648}{5}\beta_\gamma^8r^3\ . \tag{13}$$

The phase diagram of our model may now be completely determined [8] (within this RS *Ansatz*); see Fig. 1. There is a high-temperature paramagnetic phase ($m=0$, $r=0$), which in the context of non-self-avoiding membranes is the so-called crumpled phase. At a temperature

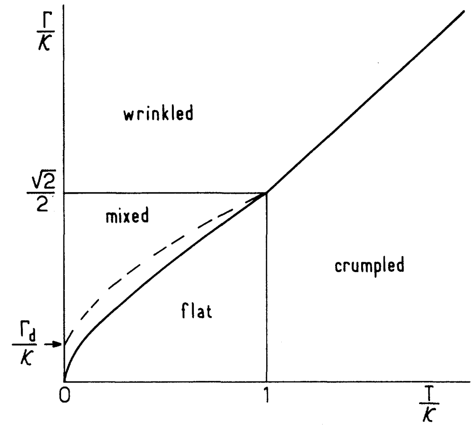

FIG. 1. Mean-field phase diagram obtained from a replica-symmetry-breaking (Parisi) *Ansatz*. The solid lines denote the boundaries between the various phase. The dashed line separating a flat from a glassy phase is obtained with a replica-symmetric *Ansatz*.

$T=\kappa$ our model exhibits a transition to a low-temperature ferromagnetic phase ($m\neq 0$) equivalent to the flat phase of membranes. Below $T_c=\sqrt{2}\Gamma$, a spin-glass phase appears, characterized by $m=0$ and $r=\tau+\tau^2/5+O(\tau^3)$ ($\tau\equiv 1-T/T_c$).

From Eq. (13) the Sherrington-Kirkpatrick line separating the ($m\neq 0$) flat phase from the ($m=0$) spin-glass phase is $\kappa/\kappa_c-1=\frac{11}{5}\tau^2$ (with $\kappa_c=T_c$). Moreover, by performing a low-temperature RS calculation of the free energy, Eq. (9), one can also determine the critical amount of disorder necessary for the destabilization of the flat phase at $T=0$: $\Gamma_d=2\kappa/\sqrt{18\pi}$. A similar phase diagram was suggested by Radzihovsky and Nelson [14] and Radzihovsky and LeDoussal [13], who considered the effect of randomness in the metric on the stability of the flat phase of membranes of inner dimensions $D>4$. Thus, in spite of the drastic simplifications made in going from the full Hamiltonian (1) to its mean-field version (2), we are able to recover the phase diagram of randomly disordered membranes and go one step beyond by being able to break the replica symmetry assumed until now.

Indeed, in analogy with other spin-glass models, one does not expect the replica-symmetric mean field to provide an accurate description of the spin-glass region of the phase diagram. It leads to negative entropies at low temperature and is unstable with respect to a mode breaking the replica symmetry [15–17].

SOLUTION BREAKING THE REPLICA SYMMETRY

The *Ansatz* that is generally believed to minimize the free energy f [Eqs. (10) and (11)] has been introduced by Parisi [15,21]. It looks for a solution $r^{\alpha\beta}$ in the subgroup of $n\times n$ block matrices of the form

$$P^{\alpha\beta}=\begin{bmatrix} \tilde{p} & p_0 & p_1 & p_1 & & \\ p_0 & \tilde{p} & p_1 & p_1 & p_2 & \cdots \\ p_1 & p_1 & \tilde{p} & p_0 & & \\ p_1 & p_1 & p_0 & \tilde{p} & & \\ & p_2 & & & \ddots & \\ & \vdots & & & & \end{bmatrix} . \tag{14}$$

In that representation, the size m_i of block i, satisfies $1=m_0<m_1<\cdots<m_k<n$. However, the idea of Parisi was to *invert* the inequalities in the limit $n\to 0$: $1=m_0>m_1>\cdots>m_k>n=0$. Representing the matrix $P^{\alpha\beta}$ by a pair $[\tilde{p},p(x)]$, with $p(x)=p_i$ for $1\geq m_i>x>m_{i+1}\geq 0$, it is easy to show [15–17] that the addition and multiplication of block matrices of the form (14) then reduces (in the limit $n\to 0$) to algebraic operations on their continuous representation:

$$A+B=C \Longleftrightarrow \begin{cases} \tilde{c}=\tilde{a}+\tilde{b} \ , \\ c(x)=a(x)+b(x) \ , \end{cases} \tag{15}$$

$$AB=C \Longleftrightarrow \begin{cases} \tilde{c}=\tilde{a}\tilde{b}-\langle ab\rangle, \quad \text{where } \langle a\rangle \equiv \int_0^x a(x)dx \ , \\ c(x)=(\tilde{b}-\langle b\rangle)a(x)+(\tilde{a}-\langle a\rangle)b(x)-\int_0^x [a(x)-a(y)][b(x)-b(y)]dy \ . \end{cases}$$

Using these relations the various terms in Eq. (11) can be evaluated. Thus

$$\frac{1}{n}\sum_{\alpha,\beta}{}'(r^{\alpha\beta})^k=-\int_0^1 dx[r(x)]^k \ ,$$
$$\frac{1}{n}\sum_{\alpha,\beta,\gamma}{}' r^{\alpha\beta}r^{\beta\gamma}r^{\gamma\alpha}=\int_0^1 dx\left[xr^3+3r\int_0^x r^2\right] . \tag{16}$$

The replica-symmetry-breaking free energy $f_{\rm RSB}$ becomes

$$\beta f_{\rm RSB}=\frac{\beta_\gamma\kappa_\gamma}{2}(1-\beta_\gamma\kappa_\gamma)m^2+\frac{\beta_\gamma^4\kappa_\gamma^4}{20}m^4-\tfrac{3}{2}\beta_\gamma^2(1+\tfrac{3}{5}\beta_\gamma^2)q^2+\tfrac{3}{2}\beta_\gamma^2(1+\beta_\gamma^2)\Delta^2$$
$$+\beta_\gamma^4\kappa_\gamma^2 m^2\int_0^1 r-\tfrac{3}{2}\beta_\gamma^2(1-2\beta_\gamma^2)\int_0^1 r^2-4\beta_\gamma^6\int_0^1 dx\left[xr^3+3r\int_0^x r^2\right]+\epsilon\beta_\gamma^8\int_0^1 r^4+\cdots \ , \tag{17}$$

where $\epsilon=6\sqrt{14}/5$ represents the joint contribution of the terms in r^4 in Eq. (11) [$\mathrm{Tr}'r^4$ and $\sum'_{\alpha,\beta}(r^{\alpha\beta})^4$] [Usually, the $\mathrm{Tr}'r^4$ term is neglected as being less destabilizing to the RS solution than $\sum'_{\alpha,\beta}(r^{\alpha\beta})^4$. Here, however, it is the most destabilizing term. It is responsible for the destabilization of the RS solution and the existence of a RSB phase. However, as far as the calculation of the Almeida-Thouless line (separating the ferromagnetic phase from a RSB phase) is concerned, the contribution of both terms can be summed up in the last term of Eq. (17). A complete solution and the subsequent evaluation of ϵ are presented in Appendix B.] The RSB spin-glass order parameter $r(x)$ is determined from the stationary condition on f_{RSB}, Eq. (10):

$$\begin{aligned}0&=\beta\frac{\delta f_{\mathrm{RSB}}}{\delta r(x)}\\&=\beta_\gamma^4\kappa_\gamma^2 m^2-6\beta_\gamma^2(\tfrac{1}{2}-\beta_\gamma^2)r(x)\\&\quad-12\beta_\gamma^6\left[xr^2+\int_0^x r^2+2r\int_x^1 r\right]+4\epsilon\beta_\gamma^8 r^3\ .\end{aligned}\tag{18}$$

Differentiating that equation once with respect to $r(x)$ yields:

$$\left[(\tfrac{1}{2}-\beta_\gamma^2)+4\beta_\gamma^4\left[xr+\int_x^1 r\right]-2\epsilon\beta_\gamma^6 r^2\right]r'=0\ .\tag{19}$$

Thus either $r=r_1=\text{const}$ or the terms in the square brackets are null. Differentiating those once more yields two more solutions: $r=r_0$ or $r=x/\epsilon\beta_\gamma^2$. The full RSB solution is the piecewise linear function

$$r(x)=\begin{cases}r_1 & \text{when } x_1<x<1\\ x/\epsilon\beta_\gamma^2 & \text{when } x_0<x<x_1=\epsilon\beta_\gamma^2 r_1\\ r_0 & \text{when } 0<x<x_0=\epsilon\beta_\gamma^2 r_0\ .\end{cases}\tag{20}$$

The two parameters r_0 and r_1 are determined by imposing the full stationary condition, Eq. (18), at $x=0$ and $x=1$. One obtains

$$\begin{aligned}r_1&=\tau+\epsilon\tau^2/4+O(\tau^3)\ ,\\ r_0&=(m^2/\epsilon)^{1/3}\ .\end{aligned}\tag{21}$$

The transition from the RS solution, $r(x)=\text{const}$ (valid in the ferromagnetic phase) to a RSB phase occurs along the Almeida-Thouless (AT) [15–17,22] line $r_0=r_1$, i.e.,

$$\tau=(m^2/\epsilon)^{1/3}=(5m^2/6\sqrt{14})^{1/3}\ .\tag{22}$$

Below that line (at lower temperatures or stronger disorder) the system "freezes" in a configuration with nonzero average magnetization, but broken ergodicity. We call that mixed state a wrinkled-flat phase. Notice that the AT line has the same dependence ($\tau\sim m^{2/3}$) as the AT line of an Ising model. This was expected [19] because of the strong coupling between the spin components mentioned previously. The fact that our random Dzyaloshinsky-Moriya model behaves in the same way [18,19] as an Ising model allows us to draw on the knowledge of the random Ising model to determine the behavior of the AT line as T and Γ go to zero:

$$T\approx\Gamma e^{-\kappa^2/2\Gamma^2}\ .\tag{23}$$

Thus one expects the existence of a wrinkled-flat phase as $T\to 0$ even for infinitesimal disorder (see Fig. 1). This is at odds with the replica-symmetric results that at $T=0$ the flat phase is stable to infinitesimal disorder. But, of course, the RS *Ansatz* is unable to describe the RSB mixed phase. Finally, the line separating that phase (with $m\neq 0$) from the RSB spin-glass phase (with $m=0$) can be established by studying the magnetic susceptibility [15–17] $\chi^{-1}=\partial^2 f_{\mathrm{RSB}}/\partial m^2$ in the spin-glass phase:

$$\begin{aligned}(\kappa\chi)^{-1}&=1-\beta\kappa+\beta\kappa\int_0^1 dx\ r(x)\\&=1-\kappa_\gamma/\sqrt{2}\ .\end{aligned}\tag{24}$$

Thus the magnetic susceptibility changes sign (i.e., $m\neq 0$) when $\kappa=\kappa_c$; see Fig. 1.

DISCUSSION

The replica-symmetry-breaking results described here were obtained within a *mean-field* approximation (valid for membranes of inner dimension $D>4$). As usual with spin glasses, it is not clear how relevant these results are in lower dimensions and, in particular, to non-self-avoiding $D=2$ membranes (even less so to real self-avoiding ones). However, one might hope that because of the existence of long-range interactions between the normals mediated by the in-plane elastic modes, the real phase diagram could be similar to these mean-field results.

One particular point that might be numerically tested is the existence of a RSB phase at $T\neq 0$ for infinitesimal disorder. As previously mentioned the RS solution of Morse *et al.* [10] predicts the existence for $D=2$ membranes of a new disordered flat phase at $T=0$. It is tempting to conjecture that a full RSB solution might extend the domain of existence of this phase to $T\neq 0$, thus encompassing the wrinkled-flat phase described above. That conjecture could be tested numerically, though the very long equilibration times of membranes and spin-glass Monte Carlo simulations might make their combination too formidable for today's computational resources [23].

From an experimental point of view, it should be possible to reconstruct the surface of a wrinkled membrane, either by freeze-etching and electron microscopy or by scanning atomic-force microscopy. This may enable one to extract the relevant microscopic information on the membrane (i.e., the orientation of the normals) and allow for a qualitative comparison with the spin-glass ideas exposed here. For example, by thermally cycling a wrinkled membrane one might be able to measure the spin-glass order parameter $r(x)$, namely, the overlap between different ground-state configurations. A nontrivial behavior of this overlap function would support the interpretation of the wrinkled state as a glassy phase.

APPENDIX A: CUMULANT EXPANSION OF THE FREE ENERGY

The free energy of one replica is given by

$$\beta f=\tfrac{3}{2}\beta_\gamma^2(\tfrac{5}{2}+q-q^2)+\frac{\beta_\gamma\kappa_\gamma m^2}{2}$$

$$+\frac{\beta_\gamma^2}{2}\sum_{\alpha,\beta}{}' T_x^{\alpha\beta}\left[T_y^{\alpha\beta}-\frac{T_x^{\alpha\beta}}{4}\right]-\frac{1}{n}\ln\langle e^X\rangle\ , \qquad \text{(A1)}$$

which is Eq. (8) rewritten with the following notations:

$$T_\mu^{\alpha\beta}\equiv \mathrm{Tr}R^{\alpha\beta}-3R_{\mu\mu}^{\alpha\beta}\ ,$$

$$X\equiv\beta_\gamma\kappa_\gamma m\sum_\alpha S_x^\alpha-\frac{3\beta_\gamma^2}{2}q\sum_\alpha (S_x^\alpha)^2$$

$$+\frac{\beta_\gamma^2}{2}\sum_{\alpha,\beta}{}'\sum_\mu T_\mu^{\alpha\beta}S_\mu^\alpha S_\mu^\beta\ ,$$

and where m and q are independent of the replica index α. Then, we make the cumulant expansion of the logarithmic term appearing in the expression of the free energy:

$$\ln\langle e^X\rangle=\langle X\rangle+\tfrac{1}{2}\langle(X-\langle X\rangle)^2\rangle+\tfrac{1}{6}\langle(X-\langle X\rangle)^3\rangle$$
$$+\tfrac{1}{24}(\langle(X-\langle X\rangle)^4\rangle-3\langle(X-\langle X\rangle)^2\rangle^2)+o(X^4)$$
$$=\langle X\rangle+\tfrac{1}{2}\langle X^2\rangle+\tfrac{1}{6}\langle X^3\rangle+\tfrac{1}{24}\langle X^4\rangle$$
$$+o(X^4)+o(n)\ . \qquad \text{(A2)}$$

Now, let us evaluate the quartic term. For this, we set

$$X\equiv a+b+c\ ,$$
$$a\equiv\beta_\gamma\kappa_\gamma m\sum_\alpha S_x^\alpha\ ,$$
$$b\equiv-\tfrac{3}{2}q\beta_\gamma^2\sum_\alpha (S_x^\alpha)^2\ , \qquad \text{(A3)}$$
$$c\equiv\tfrac{1}{2}\beta_\gamma^2\sum_{\alpha,\beta}{}'\sum_\mu T_\mu^{\alpha\beta}S_\mu^\alpha S_\mu^\beta\ .$$

Due to the integration over n spheres, with the constraint $\alpha\neq\beta$ in c, and the need to pair the spin components over each sphere, only the following terms appear in the expansion of $\langle X^4\rangle$, the other ones being equal to zero:

$$\langle X^4\rangle=\langle a^4+b^4+c^4+6(a^2b^2+b^2c^2+c^2a^2)$$
$$+4bc^3+12a^2bc\rangle\ . \qquad \text{(A4)}$$

Let us develop the example of $\langle c^4\rangle$, the other terms being easier to compute:

$$\langle c^4\rangle=\frac{\beta_\gamma^8}{16}\sum_{\mu,\nu,\rho,\sigma}\sum_{\alpha,\beta}{}'\sum_{\gamma,\delta}{}'\sum_{\epsilon,\zeta}{}'\sum_{\eta,\theta}{}' T_\mu^{\alpha\beta}T_\nu^{\gamma\delta}T_\rho^{\epsilon\zeta}T_\sigma^{\eta\theta}\langle S_\mu^\alpha S_\mu^\beta S_\nu^\gamma S_\nu^\delta S_\rho^\epsilon S_\rho^\zeta S_\sigma^\eta S_\sigma^\theta\rangle\ . \qquad \text{(A5)}$$

The pedestrian method we use to evaluate this expression consists of a decomposition of the sum over $(\alpha,\ldots,\theta)$, for fixed $(\mu,\ldots,\sigma)$, into pieces where the number of different spheres, over which we effectively integrate, is fixed:

$$\langle c^4\rangle=\frac{\beta_\gamma^8}{16}\sum_{\mu,\nu,\rho,\sigma}\Bigg[8\sum_{\alpha,\beta}{}' T_\mu^{\alpha\beta}T_\nu^{\alpha\beta}T_\rho^{\alpha\beta}T_\sigma^{\alpha\beta}\langle S_\mu S_\nu S_\rho S_\sigma\rangle^2+48\sum_{\alpha,\beta,\gamma}{}'\delta_{\mu\nu}\delta_{\rho\sigma}T_\mu^{\alpha\beta}T_\nu^{\alpha\beta}T_\rho^{\beta\gamma}T_\sigma^{\beta\gamma}\langle S_\mu S_\nu S_\rho S_\sigma\rangle$$
$$+12\sum_{\alpha,\beta,\gamma,\delta}{}' T_\mu^{\alpha\beta}T_\nu^{\alpha\beta}T_\rho^{\gamma\delta}T_\sigma^{\gamma\delta}\langle S_\mu^\alpha S_\mu^\beta S_\nu^\alpha S_\nu^\beta S_\rho^\gamma S_\rho^\delta S_\sigma^\gamma S_\sigma^\delta\rangle$$
$$+48\sum_{\alpha,\beta,\gamma,\delta}{}' T_\mu^{\alpha\beta}T_\nu^{\beta\gamma}T_\rho^{\gamma\delta}T_\sigma^{\delta\alpha}\langle S_\mu^\alpha S_\mu^\beta S_\nu^\beta S_\nu^\gamma S_\rho^\gamma S_\rho^\delta S_\sigma^\delta S_\sigma^\alpha\rangle\Bigg]\ . \qquad \text{(A6)}$$

Now, we decompose the sum over $(\mu,\ldots,\sigma)$ into sums over a fixed number of different indices. Taking into account the normalization $|S|^2=3$, we obtain

$$\langle c^4\rangle=\beta_\gamma^8\Bigg[\frac{27}{50}\sum_{\alpha,\beta}{}'\sum_{\mu,\nu}{}'(T_\mu^{\alpha\beta})^2(T_\nu^{\alpha\beta})^2+\frac{81}{50}\sum_{\alpha,\beta}{}'\sum_\mu (T_\mu^{\alpha\beta})^4+\frac{9}{5}\sum_{\alpha,\beta,\gamma}{}'\sum_{\mu,\nu}{}'(T_\mu^{\alpha\beta})^2(T_\nu^{\beta\gamma})^2$$
$$+\frac{27}{5}\sum_{\alpha,\beta,\gamma}{}'\sum_\mu (T_\mu^{\alpha\beta}T_\mu^{\beta\gamma})^2+\frac{3}{4}\sum_{\alpha,\beta,\gamma,\delta}{}'\sum_{\mu,\nu}(T_\mu^{\alpha\beta}T_\nu^{\gamma\delta})^2+3\sum_{\alpha,\beta,\gamma,\delta}{}'\sum_\mu T_\mu^{\alpha\beta}T_\mu^{\beta\gamma}T_\mu^{\gamma\delta}T_\mu^{\delta\alpha}\Bigg]\ . \qquad \text{(A7)}$$

Developing in the same way the other terms of the expansion of $\langle X^4\rangle$, we obtain the following expression for $[T^4]$:

$$[T^4]=\mathrm{Tr}'(T_x^4+2T_y^4)+\frac{4}{5}\sum_{\alpha,\beta,\gamma}{}'[(T_x^{\alpha\beta}T_x^{\beta\gamma})^2-2(T_x^{\alpha\beta}T_y^{\beta\gamma})^2+(T_y^{\alpha\beta}T_y^{\beta\gamma})^2]+\frac{1}{25}\sum_{\alpha,\beta}{}'[(T_x^{\alpha\beta})^4-32(T_x^{\alpha\beta}T_y^{\alpha\beta})^2-14(T_y^{\alpha\beta})^4]\ . \qquad \text{(A8)}$$

The same procedure applied to $[T^3]$ and $[T^2]$ gives

$$[T^3]=\mathrm{Tr}'(T_x^3+2T_y^3)-\tfrac{18}{5}q\beta_\gamma^2\mathrm{Tr}'(T_x^3-T_y^3)\ ,$$

$$[T^2]=\mathrm{Tr}'\left[\frac{T_x^2}{4}-T_xT_y\right]+\frac{\beta_\gamma^2}{2}\mathrm{Tr}(T_x^2+2T_y^2)+\beta_\gamma^2(\beta_\gamma\kappa_\gamma m)^2\sum_{\alpha,\beta,\gamma}{}' T_x^{\alpha\beta}T_x^{\beta\gamma}$$
$$-\tfrac{2}{5}\beta_\gamma^2[3q\beta_\gamma^2-(\beta_\gamma\kappa_\gamma m)^2]\mathrm{Tr}(T_x^2-T_y^2)+\tfrac{27}{175}q^2\beta_\gamma^6\mathrm{Tr}(8T_x^2-T_y^2)\ . \tag{A9}$$

If we make the change of variables $r\equiv\frac{1}{3}(T_y+T_x/2)$ and $\Delta\equiv\frac{1}{3}(T_y-T_x)$, then to leading order the term $[T^2]$ is

$$[T^2]=3(1+\beta_\gamma^2)\mathrm{Tr}'\Delta^2+3(2\beta_\gamma^2-1)\mathrm{Tr}'r^2+\cdots\ , \tag{A10}$$

which shows that r, but not Δ, undergoes a phase transition when $\beta_\gamma=1/\sqrt{2}$. Taking this into account, the relevant terms in $[T^3]$ and $[T^4]$ are

$$[T^3]=24\,\mathrm{Tr}'r^3+\cdots\ ,$$
$$[T^4]=48\sum_{\alpha,\beta,\gamma,\delta}{}' r^{\alpha\beta}r^{\beta\gamma}r^{\gamma\delta}r^{\delta\alpha}-\tfrac{144}{5}\sum_{\alpha,\beta}{}'(r^{\alpha\beta})^4+\cdots\ . \tag{A11}$$

The fourth-order cumulant expansion of the free energy is

$$\beta f=P_1+\frac{P_2}{n}\sum_{\alpha,\beta}{}' T_x^{\alpha\beta}-\frac{\beta_\gamma^2}{2n}[T^2]-\frac{\beta_\gamma^6}{6n}[T^3]-\frac{\beta_\gamma^8}{8n}[T^4]+o(n)+o(X^4)\ , \tag{A12}$$

where P_1 and P_2 are given to leading orders by

$$P_1=\frac{\beta k}{2}(1-\beta k)m^2+\frac{(\beta k m)^4}{20}-\tfrac{3}{2}\beta_\gamma^2(1+\tfrac{3}{5}\beta_\gamma^2)q^2+\cdots\ ,$$
$$P_2=-\tfrac{1}{2}(\beta_\gamma^2\kappa_\gamma m)^2+\cdots\ . \tag{A13}$$

Inserting equalities (A10), (A11), and (A13) in Eq. (A12) yields Eq. (11).

APPENDIX B: RSB SOLUTION

We proceed to the application of Parisi's *Ansatz*. The use of Eqs. (15) and (16) allows us to write the last expression of $[T^4]$, Eq. (A11), in its continuous form:

$$-\frac{1}{n}[T^4]=48\left[\int_0^1 dx\left[2r(x)\int_0^1 r+s(x)\right]^2-\tfrac{3}{5}\int_0^1 r^4+\int_0^1 dx\left[2r^2(x)\int_0^1 r^2+\int_0^1[r^2(x)-r^2(y)]^2dy\right]\right]$$
$$=\tfrac{384}{5}\int_0^1 r^4-144\left[\int_0^1 r^2\right]^2-96\int_0^1 dx\int_0^x dy[r^2(x)-r^2(y)]^2-48\int_0^1 dx\left[2r(x)\int_0^1 r+s(x)\right]^2\ . \tag{B1}$$

We have introduced the function $s(x)\equiv\int_0^x dy[r(x)-r(y)]^2$, and used the following formula derived from Eq. (15):

$$-\frac{1}{n}\sum_{\alpha,\beta,\gamma,\delta}{}' r^{\alpha\beta}r^{\beta\gamma}r^{\gamma\delta}r^{\delta\alpha}=\int_0^1 dx\left[2r(x)\int_0^1 r+s(x)\right]^2+\int_0^1 dx\left[2r^2(x)\int_0^1 r^2+\int_0^x[r^2(x)-r^2(y)]^2dy\right]\ . \tag{B2}$$

In order to find the extrema of the free energy, we equate to zero its functional derivative with respect to $r(w)$:

$$0=\beta\frac{\delta f}{\delta r(w)}=\beta_\gamma^4\kappa_\gamma^2m^2+3\beta_\gamma^2(2\beta_\gamma^2-1)r(w)-12\beta_\gamma^6\left[xr^2+\int_0^x r^2+2r\int_x^1 r\right]$$
$$+\tfrac{1}{8}\beta_\gamma^8\left[\tfrac{384}{5}r^3(w)+192r(w)\int_0^1 r^2+48\frac{\delta F}{\delta r(w)}\right]\ ,$$

$$\frac{\delta F}{\delta r(w)}\equiv 4\int_0^1 r\left[2r(w)\int_0^1 r+\int_0^1 r^2+2wr^2(w)-2r(w)\int_0^w r-2\int_w^1 r^2+2r(w)\int_w^1 r+s(w)\right]$$
$$+4\left[\int_0^1 dx\,r(x)s(x)+wr(w)s(w)-s(w)\int_0^w r-\int_w^1 rs+r(w)\int_w^1 s\right]\ . \tag{B3}$$

If we take the derivative with respect to w of this expression, we notice that $r'(w)$ factorizes, so either r is a constant over some interval, or r verifies a new integral equation. Repeating this process twice again, we find that when r is not a constant, it must verify a very simple differential equation:

$$(\tfrac{2}{5}+x^2)r''(x)+3xr'(x)=0\ ,$$

whose general solution is

$$r(x)=\frac{ax}{\sqrt{x^2+\frac{2}{5}}}+b\ .$$

The final step is to replace this expression for r in the integral equations that it verifies to determine the constants a and b. One finds $a=5/3\sqrt{35}$ and $b=0$. Near the multicritical point ($\kappa_c=T_c=\sqrt{2}\Gamma$), where the expansion leading to Eq. (11) is valid, $r\ll 1$; thus $r(x)\sim 5x/3\sqrt{14}$. The same behavior is obtained by fixing the value of ϵ in Eq. (11) to $6\sqrt{14}/5$.

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